

On the asymptotic of convex hulls of Gaussian fields*

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Abstract

We consider a Gaussian field $X = \{X_{\mathbf{t}}, \mathbf{t} \in T\}$ with values in a Banach space \mathbb{B} defined on a parametric set T equal to \mathbb{R}^m or \mathbb{Z}^m . It is supposed that the distribution \mathcal{P} of $X_{\mathbf{t}}$ is independent of t . We consider the asymptotic behavior of closed convex hulls

$$W_n = \text{conv}\{X_{\mathbf{t}}, \mathbf{t} \in T_n\}$$

where (T_n) is an increasing sequence of subsets of T and we show that under some conditions of the weak dependence with probability 1

$$\lim_{n \rightarrow \infty} \frac{1}{b_n} W_n = \mathcal{E}$$

(in the sense of Hausdorff distance), where the limit shape \mathcal{E} is the concentration ellipsoid of \mathcal{P} .

The asymptotic behavior of the mathematical expectations $Ef(W_n)$, where f is an homogeneous function is also studied.

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1 Introduction and formulation of results

Let \mathbb{B} be a separable Banach space and let $X = \{X_{\mathbf{t}}, \mathbf{t} \in T\}$ be a centered Gaussian process with values in \mathbb{B} defined on some probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. T is some parametric space, in our paper we shall consider two cases: $T = \mathbb{R}^m$ or $T = \mathbb{Z}^m$. For $\mathbf{t} = (t_1, \dots, t_m)$, $\mathbf{s} = (s_1, \dots, s_m) \in T$ (in both cases) $|\mathbf{t} - \mathbf{s}| = \max_{1 \leq k \leq m} |t_k - s_k|$. In all paper we shall assume that the marginal distributions of $X_{\mathbf{t}}$ are the same for all $\mathbf{t} \in T$ and will be denoted by \mathcal{P} . The measure \mathcal{P} is Gaussian measure on a separable Banach space, so by H we denote the reproducing Hilbert space of this measure and \mathcal{E} will stand for the ellipsoid of concentration of the measure \mathcal{P} (i.e., the unit ball in H).

Let (T_n) be an increasing sequence (this will be always understood as $T_n \subset T_{n+1}$) of subsets of T with $\nu_n \rightarrow \infty$, where, in the case of $T = \mathbb{Z}^m$, ν_n is defined as $\text{card}\{T_n\}$, while in the case of $T = \mathbb{R}^m$, $\nu_n = l^m(T_n)$, where l^m denotes the Lebesgue measure in \mathbb{R}^m . For a set $A \subset \mathbb{B}$ let us denote by $\text{conv}\{A\}$ the closed convex hull of the set A . We are interested in the limit behavior of the sequence of sets

$$W_n = \text{conv}\{X_{\mathbf{t}}, \mathbf{t} \in T_n\}.$$

This problem is interesting and important since it can be considered as the multivariate generalization of classical and deeply investigated problem on the behavior of extreme values of Gaussian processes, see, for example, [6], [1], [9] and references there. The limit behavior of sets W_n is closely related with the limit behavior of Gaussian samples, see [5], and has various interesting applications, see [8].

In [2] the case $T = \mathbb{Z}$ and $X = \{X_1, X_2, \dots\}$ being independent and identically distributed (i.i.d.) random variables with values in \mathbb{B} was studied, while in [3] the case of stationary sequences with $T = \mathbb{R}$ and $\mathbb{B} = \mathbb{R}^d$ was considered. It was stated in particular that under mild conditions with probability 1

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2 \ln n}} W_n = \mathcal{E}$$

(in the sense of Hausdorff distance), where $T_n = T \cap [0, n]$ and the limit shape \mathcal{E} is the concentration ellipsoid defined by the covariance structure of X . We generalize and complement the statements of [2] and [3].

In order to formulate our results we need some more notation. \mathbb{B}^* will stand for the conjugate space of \mathbb{B} and $\langle \cdot, \cdot \rangle$ denote the bilinear form defining

the duality between \mathbb{B} and its conjugate space. $B_r(x)$ and $S_r(x)$ denote the closed ball and the sphere, respectively, with radius r and center $x \in \mathbb{B}$, while $B_r^*(x)$ and $S_r^*(x)$ stand for corresponding sets in \mathbb{B}^* .

Since in our setting the sets W_n will be compact, we introduce the separable complete metric space $\mathcal{K}_{\mathbb{B}}$ of all nonempty compact subsets of a Banach space \mathbb{B} equipped with the Hausdorff distance $\rho_{\mathbb{B}}$:

$$\rho_{\mathbb{B}}(A, B) = \max\{\inf\{\epsilon \mid A \subset B^\epsilon\}, \inf\{\epsilon \mid B \subset A^\epsilon\}\},$$

A^ϵ is the open ϵ -neighbourhood of A . Convergence of compact sets in \mathbb{B} always will be in this metric.

Also in all paper we use the notation $b(t) = \sqrt{2 \ln(t \vee 2)}$. Our first result is in the case $T = \mathbb{Z}^m$.

Theorem 1. *Suppose that a Gaussian process X with the same marginal distributions \mathcal{P} for all t satisfies the following condition of the weak dependence*

$$\forall x^* \in \mathbb{B}^* \quad E \langle X_{\mathbf{t}}, x^* \rangle \langle X_{\mathbf{s}}, x^* \rangle \rightarrow 0, \quad \text{as } |\mathbf{t} - \mathbf{s}| \rightarrow \infty. \quad (1)$$

Then

$$\frac{1}{b(\nu_n)} W_n \xrightarrow{a.s.} \mathcal{E}, \quad \text{as } n \rightarrow \infty, \quad (2)$$

where $\xrightarrow{a.s.}$ denotes the convergence a.s. (and, as it was mentioned, in the metric $\rho_{\mathbb{B}}$).

In the continuous case ($T = \mathbb{R}^m$) we need two additional hypothesis. Now it is not sufficient to require that marginal distributions of the process X are the same, and we suppose that our process is stationary. For the subsets T_n (in discrete case it was finite sets) we assume that they are compact sets satisfying the condition

$$\forall \varepsilon > 0 \quad \lim_n \frac{l^m((\partial T_n)^\varepsilon)}{l^m(T_n)} \rightarrow 0, \quad (3)$$

where ∂T_n stands for the boundary of T_n .

Theorem 2. *Suppose that the process $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}^m\}$ is stationary and the conditions (1) and (3) are fulfilled. Then $W_n \in \mathcal{K}_{\mathbb{B}}$ a.s. and the relation (2) takes place.*

Having the results on the convergence a.s. (and, therefore, in distribution), we can easily obtain, as in [2], the convergence of mean values for various functionals of these distributions.

Let $f : \mathcal{K}_{\mathbb{B}} \rightarrow \mathbb{R}$ be a continuous non-negative increasing homogeneous function of degree p , that is

$$\begin{aligned} f(A) &\geq 0 \quad \forall A \in \mathcal{K}_{\mathbb{B}}; \\ f(A_1) &\leq f(A_2) \quad \forall A_1 \subset A_2, \quad A_1, A_2 \in \mathcal{K}_{\mathbb{B}}; \\ f(cA) &= c^p f(A), \quad \forall c \geq 0, \quad \forall A \in \mathcal{K}_{\mathbb{B}}. \end{aligned}$$

Theorem 3. *Let f be a homogeneous function of degree p with the properties described above. Suppose additionally that there exists a constant C such that for all $A \in \mathcal{K}_{\mathbb{B}}$*

$$f(A) \leq C[d(A)]^p, \quad (4)$$

where $d(A) = \sup_{x,y \in A} \|x - y\|$ is the diameter of A . Then, under hypothesis of Theorem 1 or 2 for all $a > 0$

$$E \exp \left\{ a f^{\frac{2}{p}} \left(\frac{1}{b(\nu_n)} W_n \right) \right\} \longrightarrow \exp \left\{ a f^{\frac{2}{p}}(W) \right\}.$$

Corollary 4. *Let f be a function with the properties described above. Then, under hypothesis of Theorem 1 or 2 for all $m > 0$*

$$E f^m \left(\frac{1}{b(\nu_n)} W_n \right) \rightarrow f^m(W).$$

This theorem and corollary give in particular the asymptotic behavior for mean values of all reasonable geometrical characteristics of W_n (such as diameter or volume and surface measure in the case of finite-dimensional \mathbb{B}).

2 Auxiliary lemmas

The first lemmas are about compact sets in \mathbb{B} .

Lemma 5. *If $A \in \mathcal{K}_{\mathbb{B}}$, then $\text{conv}(A) \in \mathcal{K}_{\mathbb{B}}$ and the mapping $\text{conv} : \mathcal{K}_{\mathbb{B}} \longrightarrow \mathcal{K}_{\mathbb{B}}, \quad A \rightarrow \text{conv}(A)$, is 1-Lipshitz :*

$$\rho_{\mathbb{B}}(\text{conv}(A), \text{conv}(B)) \leq \rho_{\mathbb{B}}(A, B).$$

The proof of the lemma is elementary and is left for a reader.

Lemma 6. *Suppose that $A_n, A \in \mathcal{K}_{\mathbb{B}}$ are such that for some sequence $(\varepsilon_n) \downarrow 0$*

$$A_n \subset A^{\varepsilon_n}, \quad \forall n.$$

Then (A_n) is relatively compact in $\mathcal{K}_{\mathbb{B}}$.

Proof. Let $S_n = \{x_{n,j}, j = 1, \dots, k_n\}$ be a ε_n -net for the set A , then it will be $2\varepsilon_n$ -net for the set A^{ε_n} . Let us consider the set

$$\tilde{A}_n = (A \cap A_n) \cup \{x_{n,j} : B_{2\varepsilon_n}(x_{n,j}) \cap A \neq \emptyset\}.$$

From the construction of the sets \tilde{A}_n we have that \tilde{A}_n are compact sets and $\tilde{A}_n \subset A$ for all n . It is known that if a sequence of compact sets is inside of one fixed compact sets, then this sequence is relatively compact (see [10], Th.1.8.4., for finite dimensional case; for Banach spaces we have no relevant reference, but the proof is analogous). Again, from the construction of the sets \tilde{A}_n we have the following relations

$$A_n \subset \tilde{A}_n^{2\varepsilon_n}, \quad \tilde{A}_n \subset A^{2\varepsilon_n},$$

whence it follows that $\rho_{\mathbb{B}}(A_n, \tilde{A}_n) \leq 2\varepsilon_n$. Since the sequence \tilde{A}_n is relatively compact, the same property has the sequence A_n , too. The lemma is proved. \square

Lemma 7. *If $(A_n), A_n \in \mathcal{K}_{\mathbb{B}}$, is relatively compact, then the sequence $\{\text{conv}(A_n)\}$ is relatively compact, too.*

This fact follows directly from Lemmas 5 and 6.

Lemma 8. *If $A = \bigcap_n A_n$, where $A_n \in \mathcal{K}_{\mathbb{B}}$, and $(A_n) \downarrow$, then $A_n \rightarrow A$.*

Proof. Let's assume the opposite. Then for some $\delta > 0$ there exists a subsequence (n') , for which $\rho_{\mathbb{B}}(A, A_{n'}) > \delta \quad \forall n'$. Without restriction of a generality we can suppose that $(n') = (n)$. Then, as A_n can not be a subset of A^δ , we can find $x_n \in A_n$ such that $d(x_n, A) > \delta$ (here $d(x, A)$ stands for a distance from a point x and a set A in a Banach space \mathbb{B}). As $A_n \subset A_1$,

and A_1 is compact, it is possible to choose a subsequence (n_k) , for which $x_{n_k} \rightarrow x_0$. It is clear that for the limit point x_0 we will have $d(x_0, A) \geq \delta$. On the other hand, for each m and for all sufficiently large k , $x_{n_k} \in A_m$, which means that $x_0 \in A_m$, $\forall m$. Hence, x_0 must belong to A in contradiction to the previous conclusion. \square

Lemma 9. *Under conditions of Theorem 1 the sequence $\left\{ \frac{1}{b(\nu_n)} W_n \right\}$ is relatively compact a.s.*

Proof. Let's show that with probability 1 compact sets

$$K_n = \left\{ \frac{1}{b(\nu_n)} X_{\mathbf{k}}, \mathbf{k} \in T_n \right\}$$

form a relatively compact sequence in $\mathcal{K}_{\mathbb{B}}$. Then, due to Lemma 8, we get the result.

Let us renumber r.v. $X_{\mathbf{k}}$ with the indices \mathbf{k} from $\cup_n T_n$ as follows: at first somehow (but in a row) let's enumerate the random variables with indices lying in T_1 (there will be ν_1 of them), then will add the indices corresponding to random variables from $T_2 \setminus T_1$, and so on. The sequence obtained in this way we will denote by $\{Z_n\}$.

As r.v. Z_k have the same distribution, it is possible to use the first part of Theorem 1 from [5] (in its proof the assumption of independence isn't used), which gives a.s. convergence

$$\max_{k \leq n} d(Z_k, b(n)\mathcal{E}) \rightarrow 0, \quad n \rightarrow \infty.$$

It means that

$$K_n \subset \mathcal{E}^{2\varepsilon_n},$$

where a.s.

$$\varepsilon_n = \max_{k \leq b(\nu_n)} \left\{ d \left(\frac{Z_k}{b(\nu_n)}, \mathcal{E} \right) \right\} \rightarrow 0.$$

As \mathcal{E} is compact, we conclude the proof applying Lemma 6. \square

Lemma 10. *Let (ξ_n) be a real-valued Gaussian centered sequence with $\text{Var}(\xi_n) = \sigma^2 \quad \forall n$. Let*

$$c = \liminf_n \left\{ \frac{1}{b(n)} \max_{k \leq n} \{\xi_k\} \right\}.$$

Suppose that

$$r = \sup_{n \neq l} \frac{|E\xi_n \xi_l|}{\sigma^2} < 1/2.$$

Then

$$\sigma(\sqrt{1-r} - \sqrt{r}) \leq c \leq \sigma.$$

Proof. The upper bound $c \leq \sigma$ is the well-known fact (see i.e. Lemma 14 below), and for the proof of the lower bound we introduce independent standard Gaussian random variables η and $\zeta_k, k \geq 1$ and define

$$\tilde{\xi}_n = \sigma\sqrt{1-r}\zeta_n + \sigma\sqrt{r}\eta.$$

Then

$$\text{Var}(\tilde{\xi}_n) = \sigma^2 \quad \text{and} \quad E\tilde{\xi}_n \tilde{\xi}_m = \sigma^2 r \geq E\xi_n \xi_m, \quad \forall n, m.$$

Therefore, from Slepian lemma (see Corollary 3.12 in [7]) it follows that

$$\mathbf{P} \left\{ \max_{k \leq n} \xi_k \leq l \right\} \leq \mathbf{P} \left\{ \max_{k \leq n} \tilde{\xi}_k \leq l \right\}, \quad \forall l.$$

Denoting

$$Z_n = \frac{1}{b_n} \max_{k \leq n} \xi_k, \quad \tilde{Z}_n = \frac{1}{b_n} \max_{k \leq n} \tilde{\xi}_k,$$

and taking $l = \sigma s b_n$ with $s < \sqrt{1-r} - \sqrt{r}$ and $b_n = b(n)$, we have

$$\mathbf{P} \{Z_n \leq \sigma s\} \leq \mathbf{P} \{\tilde{Z}_n \leq \sigma s\}. \quad (5)$$

It remains to prove

$$\sum_n \mathbf{P} \{Z_n \leq \sigma s\} < \infty, \quad (6)$$

since then by Borel-Cantelli lemma it will follow that

$$\liminf_n Z_n \geq \sigma s \quad \text{a.s.}$$

Taking into account the relation (5) it is sufficient to prove that

$$\sum_n \mathbf{P} \left\{ \tilde{Z}_n \leq \sigma s \right\} < \infty, \text{ for all } s < \sqrt{1-r} - \sqrt{r}. \quad (7)$$

For this aim we must to show that

$$\sum_n J_n(s) < \infty, \text{ for all } s < \sqrt{1-r} - \sqrt{r}, \quad (8)$$

where

$$J_n \equiv J_n(s) = \int_R \Phi^n \left(\frac{sb_n - \sqrt{r}t}{\sqrt{1-r}} \right) \varphi(t) dt.$$

Here φ and Φ are the density function and distribution function, respectively, of a standard normal random variable. To simplify the notation, we denote

$$d = \frac{s}{\sqrt{1-r}} < 1, \quad a = \sqrt{\frac{r}{1-r}},$$

Then we can write

$$J_n = \int_R \Phi^n (db_n - at) \varphi(t) dt = a^{-1} \int_R \Phi^n (db_n - y) \varphi(y/a) dy.$$

Let us take a positive real number ε , which will be chosen later and write

$$J_n = I_{1,n} + I_{2,n},$$

where $I_{1,n}$ and $I_{2,n}$ are corresponding integrals over the intervals $(-\infty, -\varepsilon b_n)$ and $(-\varepsilon b_n, \infty)$. In the first interval we simply estimate $\Phi^n (db_n - y) \leq 1$ and we get

$$I_{1,n} \leq \int_{\varepsilon b_n a^{-1}}^{\infty} \varphi(t) dt \leq \frac{Ca}{\varepsilon b_n} \exp(-\varepsilon^2 b_n^2 a^{-2}/2) = \frac{C}{\varepsilon_2 b_n} n^{-\varepsilon^2 a^{-2}}.$$

Here and in what follows C stands for an absolute constant, not necessary the same in different places. If we chose ε satisfying condition

$$\varepsilon > a = \sqrt{\frac{r}{1-r}} \quad (9)$$

then we get

$$\sum_n I_{1,n} < \infty. \quad (10)$$

Let us note that $\Phi^n(db_n - y)$ is the decreasing function of y , therefore

$$I_{2,n} \leq \Phi^n((d + \varepsilon)b_n). \quad (11)$$

We have

$$\Phi^n((d + \varepsilon)b_n) = \left(1 - \int_{(d+\varepsilon)b_n}^{\infty} \varphi(t)dt\right)^n.$$

Since $1 - \Phi(z) \sim z^{-1}\varphi(z)$ for $z \rightarrow \infty$, there exists a constant $c_1 > 0$ such that for sufficiently large z , $1 - \Phi(z) > c_1 z^{-1}\varphi(z)$, therefore, for sufficiently large n

$$\begin{aligned} \Phi^n((d + \varepsilon)b_n) &\leq (1 - (1 - \Phi((d + \varepsilon)b_n)))^n \\ &\leq \exp\{-n(1 - (1 - \Phi((d + \varepsilon)b_n)))\} \\ &\leq \exp\left(-\frac{c_1 n^{1-(d+\varepsilon)^2}}{(d + \varepsilon)b_n}\right). \end{aligned} \quad (12)$$

From (11) and (12) we obtain

$$I_{2,n} \leq \Phi^n((d + \varepsilon)b_n) \leq \exp\left(-\frac{c_1 n^{1-(d+\varepsilon)^2}}{(d + \varepsilon)b_n}\right).$$

Now, if we chose ε satisfying condition

$$\varepsilon < 1 - d = 1 - \frac{s}{\sqrt{1-r}}, \quad (13)$$

then

$$\sum_n I_{2,n} \leq \sum_n \exp\left(-\frac{c_1 n^{1-(d+\varepsilon)^2}}{(d + \varepsilon)b_n}\right) < \infty. \quad (14)$$

It remains to note that due to the condition $s < \sqrt{1-r} - \sqrt{r}$ it is possible to choose ε , satisfying both conditions (9) and (13), since

$$\sqrt{\frac{r}{1-r}} < 1 - \frac{s}{\sqrt{1-r}}.$$

Estimates (10) and (14) prove (8). The lemma is proved. \square

Remark 11. At first we were sure that only simple estimates which we had used do not allow to prove stronger statement, namely, under condition that $r < 1$

$$\sigma\sqrt{1-r} \leq c \leq \sigma. \quad (15)$$

It turned out that even exact investigation of the integrand function

$$\Phi^n \left(\frac{sb_n - \sqrt{r}}{\sqrt{1-r}} \right) \varphi(t)$$

does not allow to achieve this goal. Contrary, it is possible to show (we do not provide these calculations since they are rather lengthy) that for $\sqrt{1-r} - \sqrt{r} \leq s < \sqrt{1-r}$ the series in (8) diverges. But since the divergence of this series does not imply the divergence of series in (6), the question if the above stated strengthening (15) of the lemma is possible remains open.

Lemma 12. *Let $(\xi_{\mathbf{k}})$, $\mathbf{k} \in \mathbb{Z}^m$, be a real-valued Gaussian centered field with $\text{Var}(\xi_{\mathbf{k}}) = \sigma^2 \quad \forall \mathbf{k}$ and $r_{\mathbf{k},\mathbf{l}} = E\xi_{\mathbf{k}}\xi_{\mathbf{l}} \rightarrow 0$ as $|\mathbf{k} - \mathbf{l}| \rightarrow \infty$. Let (T_n) be an increasing sequence of subsets of \mathbb{Z}^m with $\nu_n = \text{card}\{T_n\} \rightarrow \infty$.*

Then

$$Z_n = \frac{1}{b(\nu_n)} \max_{\mathbf{k} \in T_n} \{\xi_{\mathbf{k}}\} \xrightarrow{a.s.} \sigma. \quad (16)$$

Proof. Fix $\varepsilon \in (0, 1/2)$. By condition there exists $a > 0$ such that $|r_{\mathbf{k},\mathbf{l}}| < \varepsilon\sigma^2$ if $|\mathbf{k} - \mathbf{l}| \geq a$. We will show that it is possible to find an increasing sequence (\tilde{T}_n) of subsets of \mathbb{Z}^m with the following properties:

1. $\tilde{T}_n \subset T_n$;
2. $T_n \subset (\tilde{T}_n)^a$;
3. $\forall \mathbf{k}, \mathbf{l} \in \cup_n \tilde{T}_n, \mathbf{k} \neq \mathbf{l}$, we have $|\mathbf{k} - \mathbf{l}| \geq a$.

From 1.–3. it follows that

$$\tilde{\nu}_n := \text{card}\{\tilde{T}_n\} \leq \nu_n \leq (2a)^m \tilde{\nu}_n.$$

Therefore, $b(\tilde{\nu}_n) \sim b(\nu_n)$, and we have a.s.

$$\liminf_n Z_n \geq \liminf_n \frac{1}{b(\nu_n)} \max_{\mathbf{k} \in \tilde{T}_n} \{\xi_{\mathbf{k}}\} \geq \sigma\varphi(\varepsilon),$$

where $\varphi(r) = \sqrt{1-r} - \sqrt{r}$, by Lemma 10.

As $\varphi(r) \rightarrow 1$ when $r \rightarrow 0$, we deduce that a.s. $\liminf_n Z_n \geq \sigma$. The opposite inequality $\limsup_n Z_n \leq \sigma$ being well known, we arrive to (16).

Now we provide the construction of the sequence (\tilde{T}_n) . For a finite subset B of \mathbb{Z}^m denote by $\mathcal{L}_a(B)$ the family $\{E, E \subset B\}$ of all subsets of B such that $\forall \mathbf{k}, \mathbf{l} \in E, \mathbf{k} \neq \mathbf{l}$, we have $|\mathbf{k} - \mathbf{l}| \geq a$. If $\mathcal{L}_a(B)$ is not empty, let B_a be one of its elements of maximal cardinality. If $\mathcal{L}_a(B)$ is empty, we use the notation B_a for arbitrary chosen singleton $\{\mathbf{k}\} \subset B$. In any case it is clear that

$$B_a \subset B \subset (B_a)^a,$$

which gives the inequalities

$$\text{card}(B_a) \leq \text{card}(B) \leq (2a)^m \text{card}(B_a).$$

We define our sequence by induction. We set $\tilde{T}_1 = (T_1)_a$. When \tilde{T}_n is defined, then \tilde{T}_{n+1} is equal to \tilde{T}_n , if $T_{n+1} \subset (T_n)^a$, and \tilde{T}_{n+1} is equal to $\tilde{T}_n \cup (T_{n+1} \setminus (T_n)^a)_a$ in the case when $T_{n+1} \setminus (T_n)^a \neq \emptyset$.

It is easy to see that the properties 1.–3. are fulfilled. Therefore the lemma is proved. \square

In the sequel we shall need the notion of a support function. The function $\mathcal{M}_A(\theta)$, $\theta \in S_1^*(0)$, defined by the relation

$$\mathcal{M}_A(\theta) = \sup_{x \in A} \langle x, \theta \rangle, \quad \theta \in S_1^*(0),$$

is called a *support function* of a set $A \in \mathcal{K}^d$.

A compact convex set A is characterized by its support function since

$$A = \bigcap_{\theta \in S^*(0,1)} \{u \in \mathbb{B}; \langle u, \theta \rangle \leq \mathcal{M}_A(\theta)\}.$$

It follows easily from definition that \mathcal{M}_A is 1-Lipshitz and that

$$\rho_{\mathbb{B}}(A, B) = \sup_{\|\theta\|=1} |\mathcal{M}_A(\theta) - \mathcal{M}_B(\theta)|.$$

Lemma 13. *Let $(B_n)_{n \geq 0}$ be a sequence of random convex elements in $\mathcal{K}_{\mathbb{B}}$. Assume that (B_n) is a.s. relatively compact. Assume also that there exists a (deterministic) function $\varphi : S_1^*(0) \rightarrow \mathbb{R}$ such that, for all $\theta \in S_1^*(0)$,*

$$\mathcal{M}_{B_n}(\theta) \rightarrow \varphi(\theta) \quad \text{a.s., as } n \rightarrow +\infty.$$

Then φ is the support function of a set $A \in \mathcal{K}_{\mathbb{B}}$ and

$$B_n \rightarrow A \quad \text{a.s., as } n \rightarrow +\infty.$$

Proof. Let $\Omega_1, \mathbb{P}(\Omega_1) = 1$, be a subset of ω for which the sequence (B_n) is relatively compact. Let D be a countable dense subset of $S_1^*(0)$ and $\Omega_2, \mathbb{P}(\Omega_2) = 1$, be a subset of ω for which $\mathcal{M}_{B_n}(\theta) \rightarrow \varphi(\theta)$. Fix ω from $\Omega_1 \cap \Omega_2$. Let $A \in \mathcal{K}_{\mathbb{B}}$ be a limit point of the sequence $(B_n)_{n \geq 1}$. We denote by $(m_n)_{n \geq 1}$ an increasing sequence such that $B_{m_n} \rightarrow A$. Then, for all $\theta \in D$, $\mathcal{M}_{B_{m_n}}(\theta) \rightarrow \mathcal{M}_A(\theta)$, as $n \rightarrow +\infty$. At the same time $\mathcal{M}_{B_{m_n}}(\theta) \rightarrow \varphi(\theta)$. Using uniqueness of the limit, we obtain the equality $\mathcal{M}_A = \varphi$ a.s. on D . If A' is another limit point of the sequence $(B_n)_{n \geq 1}$, we have also $\mathcal{M}_{A'} = \varphi$ on D . Consequently $\mathcal{M}_A = \mathcal{M}_{A'}$ on D and by continuity, the equality holds on $S_1^*(0)$. Finally, (B_n) has a unique limit point and $B_n \rightarrow A$ in $\mathcal{K}_{\mathbb{B}}$ almost surely as $n \rightarrow +\infty$. Since for each $\omega \in \Omega_1 \cap \Omega_2$ we get the same deterministic support function of the set A we get that this limit point A is deterministic and its support function $\mathcal{M}_A = \varphi$. \square

We will need also the following general result, dealing with the maximum of sub-Gaussian random variables (see i.e. [2], Lemmas 1 and 3 therein).

Lemma 14. *Let $(Y_n)_{n \geq 0}$ be a sequence of identically distributed random variables such that for some $\zeta > 0$*

$$E[e^{\gamma Y_0^2}] < \infty, \quad \text{for all } \gamma < \frac{1}{2\zeta^2}.$$

Let

$$Z_n = \frac{1}{\sqrt{2 \ln n}} \max\{Y_1, \dots, Y_n\}.$$

Then,

$$\limsup_{n \rightarrow +\infty} Z_n \leq \zeta \quad \text{a.s.,}$$

and for any $a > 0$,

$$\limsup_n E \exp \{a Z_n^2\} < \infty.$$

Note that Lemmas 1 and 3 in [2] are stated for independent random variables, but it is clear from the proof that the assumption of independence is unnecessary.

3 Proofs of Theorems

Proof of Theorem 1. It is easy to see that

$$\mathcal{M}_{\mathcal{E}}(\theta) = \sqrt{E \langle X_{\mathbf{k}}, \theta \rangle^2}, \quad \theta \in S_1^*(0).$$

Due to Lemmas 9 and 13 it is sufficient to show that $\forall \theta \in S_1^*(0)$

$$\mathcal{M}_n(\theta) \xrightarrow{a.s.} \mathcal{M}_{\mathcal{E}}(\theta), \quad n \rightarrow \infty, \quad (17)$$

where \mathcal{M}_n is the support function of $Z_n = (b(\nu_n))^{-1} W_n$. As

$$\mathcal{M}_n(\theta) = \frac{1}{b(\nu_n)} \max_{\mathbf{k} \in T_n} \langle X_{\mathbf{k}}, \theta \rangle,$$

and

$$E \langle X_{\mathbf{k}}, \theta \rangle \langle X_{\mathbf{l}}, \theta \rangle \rightarrow 0 \quad \text{when } |\mathbf{k} - \mathbf{l}| \rightarrow \infty,$$

we get (17) by Lemma 12. □

Proof of Theorem 2. For $h > 0$, let us denote by $C_{\mathbf{k},h}$ the cube $[\mathbf{k}h, (\mathbf{k} + \mathbf{1})h]$ and

$$G_n = \{\mathbf{k} : C_{\mathbf{k},h} \cap T_n \neq \emptyset\},$$

$$T_{n,h} = \bigcup_{\mathbf{k} \in G_n} C_{\mathbf{k},h}.$$

It is clear that $T_n \subset T_{n,h}$ and

$$l^m(T_{n,h} \setminus T_n) \leq l^m((\partial T_n)^{2\sqrt{m}h}),$$

and also

$$\text{card} \{G_n\} h^m = l^m(T_{n,h}).$$

It follows from (3) that $\tilde{\nu}_n = \text{card} \{G_n\} \sim \nu_n h^{-m}$, therefore, $b(\tilde{\nu}_n) \sim b(\nu_n)$.

Let $Z_n = \{X_{\mathbf{t}}, \mathbf{t} \in T_n\}$, $Z_{n,h} = \{X_{\mathbf{k}h}, \mathbf{k} \in G_n\}$. Then

$$Z_{n,h} \subset (Z_n)^{d_n}, \quad Z_n \subset (Z_{n,h})^{d_n}, \quad (18)$$

where $d_n = \max_{\mathbf{k} \in G_n} \zeta_{\mathbf{k}}$, and $\zeta_{\mathbf{k}} = \sup \{|X_{\mathbf{t}} - X_{\mathbf{k}h}|, \mathbf{t} \in C_{\mathbf{k},h}\}$.

Relations (18) mean that

$$\rho_{\mathbb{B}}(Z_n, Z_{n,h}) \leq d_n. \quad (19)$$

By stationarity the random variables $\zeta_{\mathbf{k}}$ are identically distributed. From the continuity of X it follows that $\zeta_{\mathbf{k}} < \infty$ a.s. As $\zeta_{\mathbf{k}}$ is the supremum of Gaussian random variables with variances less than $2\sigma^2(h)$, where

$$\sigma^2(h) = \sup_{|\mathbf{t}-\mathbf{s}| \leq h} E|X_{\mathbf{t}} - X_{\mathbf{s}}|^2,$$

then, according to the Fernique theorem from ([4]), for all $a < \frac{1}{4\sigma^2(h)}$,

$$M(a) = E \exp\{a\zeta_{\mathbf{k}}^2\} < \infty.$$

Now, due to Lemma 14, we have a.s.

$$\limsup_n \{d_n\} \leq \sigma(h). \quad (20)$$

By Theorem 1 for any $h > 0$

$$\frac{1}{b(\nu_n)} Z_{n,h} \xrightarrow{a.s.} \mathcal{E}. \quad (21)$$

From (19) and (20) we have a.s.

$$\limsup_n \rho_{\mathbb{B}} \left(\frac{1}{b(\nu_n)} Z_n, \frac{1}{b(\nu_n)} Z_{n,h} \right) \leq \sigma(h).$$

Hence for each h a.s.

$$\limsup_n \rho_{\mathbb{B}} \left(\frac{1}{b(\nu_n)} Z_n, \mathcal{E} \right) \leq \sigma(h).$$

Due to the continuity we have that $\sigma(h) \rightarrow 0$, if $h \rightarrow 0$, therefore, finally we get

$$\limsup_n \rho_{\mathbb{B}} \left(\frac{1}{b(\nu_n)} Z_n, \mathcal{E} \right) \xrightarrow{a.s.} 0.$$

The theorem is proved. \square

Proof of Theorem 3. Due to the continuity of f and the convergence (2) the result of Theorem 3 will follow from the uniform integrability of the family $\left\{ f \left(\frac{W_n}{b(\nu_n)} \right) \right\}$.

Due to the condition (4) we have

$$f \left(\frac{W_n}{b(\nu_n)} \right) \leq C \left(\frac{D_n}{b(\nu_n)} \right)^p,$$

where $D_n = \max_{\mathbf{k}, \mathbf{l} \in T_n} \|X_{\mathbf{k}} - X_{\mathbf{l}}\| \leq 2 \max_{\mathbf{k} \in T_n} \|X_{\mathbf{k}}\|$. Hence it is sufficient to state that for all $a > 0$

$$\sup_n E \exp \left\{ a \left(\frac{D_n}{b(\nu_n)} \right)^2 \right\} < \infty. \quad (22)$$

The latter relation follows directly from Lemma 14, and the theorem is proved. \square

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